

Due For

4.8 - Row Space, Column Space, and Null Space

Definition: A homogeneous linear system $Ax = 0$ and an associated nonhomogeneous system $Ax = b$ are **corresponding linear systems**.

#28 Find a general solution of the system, and use that solution to find a general solution of the associated homogeneous system and a particular solution of the given system.

inputs: \mathbb{R}^4 , outputs: \mathbb{R}^3

$$\begin{bmatrix} 9 & -3 & 5 & 6 \\ 6 & -2 & 3 & 1 \\ 3 & -1 & 3 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -8 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 9 & -3 & 5 & 6 & 4 \\ 6 & -2 & 3 & 1 & 5 \\ 3 & -1 & 3 & 14 & -8 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & -1/3 & 0 & -13/3 & 13/3 \\ 0 & 0 & 1 & 9 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

If the system were homogeneous, we would have $x_1 = \frac{1}{3}x_2 + \frac{13}{3}x_4$ and $x_3 = -9x_4$

Letting $x_2 = s$ and $x_4 = t$, we get

$$\vec{x} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} \frac{13}{3} \\ 0 \\ -9 \\ 1 \end{bmatrix} t$$

With the constants from b , we get

$$x_1 = \frac{1}{3}x_2 + \frac{13}{3}x_4 + \frac{13}{3}, \quad x_3 = -9x_4 - 7, \quad \text{so}$$

$$\vec{x} = \begin{bmatrix} 1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 13/3 \\ 0 \\ -9 \\ 1 \end{bmatrix} t + \begin{bmatrix} 13/3 \\ 0 \\ -7 \\ 0 \end{bmatrix}$$

General solution
to the homogeneous
system

particular solution
to the nonhomogeneous
system, \vec{x}_p .

Definition: The solution set of a homogeneous linear system $Ax = 0$ is called the **general solution**. A constant vector \vec{x}_0 in the solution set of a consistent linear system $Ax = b$ is a **particular solution**.

Theorem 4.8.2 If \vec{x}_0 is any solution of a consistent linear system $Ax = b$, and if $S = \{v_1, v_2, \dots, v_k\}$ is a basis for the null space of A , then every solution of $Ax = b$ can be expressed in the form $x = \vec{x}_0 + c_1 v_1 + c_2 v_2 + \dots + c_k v_k$. Conversely, for all choices of scalars c_1, c_2, \dots, c_k , the vector x in this formula is a solution of $Ax = b$.

solution space to
the homogeneous
system

Note that the solution space of $A\vec{x} = \vec{b}$ can be considered a translation of the solution space of $A\vec{x} = \vec{0}$ (null space) by the particular solution vector.

Pf (Thm 4.8.2): Let $W = \text{span}(S) = \text{null}(A)$,

\vec{x}_0 be a solution of $A\vec{x} = \vec{b}$, and

$\vec{x} \in \vec{x}_0 + W$, the set of vectors obtained by adding \vec{x}_0 to all vectors in W .

(\Rightarrow) Let \vec{x} be any solution of $A\vec{x} = \vec{b}$ and let $\vec{w} \in W$, so that $A\vec{w} = \vec{0}$.

For $\vec{w} = \vec{x} - \vec{x}_0$, we have

$$\begin{aligned} A\vec{w} &= A(\vec{x} - \vec{x}_0) = A\vec{x} - A\vec{x}_0 \\ &= \vec{b} - \vec{b} = \vec{0}, \quad / \quad c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n \end{aligned}$$

$$\text{and } \vec{x} = \vec{x}_0 + \vec{w} = \vec{x}_0 + \sum_{i=1}^k c_i \vec{v}_i.$$

(\Leftarrow) Suppose $\vec{x} \in \vec{x}_0 + W$. Then $\vec{x} = \vec{x}_0 + \vec{w}$ for some $\vec{w} \in W$.

$$\begin{aligned} A\vec{x} &= A(\vec{x}_0 + \vec{w}) = A\vec{x}_0 + A\vec{w} \\ &= \vec{b} + \vec{0} = \vec{b}. \end{aligned}$$

Thus \vec{x} is a solution to $A\vec{x} = \vec{b}$ ✓

The vectors in the general solution are linearly independent and so form a basis for the null space of the homogeneous system, denoted by \vec{X}_h .

Definition: For an $m \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ the vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \ a_{12} \ \dots \ a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \ a_{22} \ \dots \ a_{2n}] \\ &\vdots \\ \mathbf{r}_m &= [a_{m1} \ a_{m2} \ \dots \ a_{mn}] \end{aligned}$$

in R^n that are formed from the rows of A are called the **row vectors** of A , and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m that are formed from the columns of A are called the **column vectors** of A .

$$\Rightarrow T_A: R^n \rightarrow R^m$$

Definition: If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is denoted by row (A) and is called the **row space** of A , and the subspace of R^m spanned by the column of vectors of A is denoted by col (A) and is called the **column space** of A . The solution space of the homogeneous system of equations $Ax = \mathbf{0}$, which is a subspace of R^n , is denoted by null (A) and is called the **null space** of A .

domain of T_A

$$A = [\vec{c}_1 \ \vec{c}_2 \ \dots \ \vec{c}_n] \Rightarrow A\vec{x} = x_1\vec{c}_1 + x_2\vec{c}_2 + \dots + x_n\vec{c}_n = \vec{b}$$

Theorem 4.8.1 A system of linear equations $Ax = b$ is consistent if and only if b is in the column space of A .

Theorem 4.8.3

\nleftrightarrow If $A \xrightarrow[\text{ops}]{(\text{elem. row})} B$, then $\text{row}(A) = \text{row}(B)$

a) Row equivalent matrices have the same row space.

b) Row equivalent matrices have the same null space.

$A \rightarrow \text{net} \rightarrow \text{solution}$

Theorem 4.8.4 If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R , and the column vectors with the leading 1's of the row vectors form a basis for the column space of R .
Row reduce and look for pivots

#9 Find bases for the null space and row space of A .

a. $A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$ Solution space for homog. system

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} x_1 = 16x_3 \\ x_2 = 19x_3 \end{array}$$

$\vec{x} = \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} t$ so a basis for

null(A) is $\left\{ \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix} \right\}$.

Basis for row(A): $\left\{ [1 \ 0 \ -16], [0 \ 1 \ -19] \right\}$

Pf Thm 4.8.3: (a) Matrices $A \stackrel{r}{\sim} B$ are row equivalent if one can be obtained from the other by a series of elementary row operations. Since vector spaces are closed under addition and scalar multiplication, $\text{row}(B) \subseteq \text{row}(A)$ and $\text{row}(A) \subseteq \text{row}(B)$.

(b) Elementary row operations do not change the solution of a homogeneous system. Thus if $A \stackrel{r}{\sim} B$ are row equivalent, then the solution to $A\vec{x} = \vec{0}$ is the same as that $B\vec{x} = \vec{0}$ and hence $A \stackrel{r}{\sim} B$ have the same null space.

Note: In general, elementary row operations do not preserve a column space.

$$\text{b. } A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = \frac{1}{2}x_3 \\ x_2 \text{ is free} \end{array}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1/2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s$$

Basis for null space of A : $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Basis for row(A) : $\{ [2 \ 0 \ -1] \}$

Thm 4.8.3 & 4.8.4 give us

$A \xrightarrow[\text{ops}]{\text{(Elem.) row}} R \rightarrow$ basis for row(R) and row(A)
 \rightarrow basis for col(R)
 (but not col(A))

Theorem 4.8.5 If A and B are row equivalent matrices, then:

- A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
- A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B .

This means that elem. row ops do not change the dimension of a column space.

$$A \xrightarrow[\text{ops}]{\text{(elem.) row}} R$$

use columns of R that contain leading 1s to identify columns of A that form a basis for $\text{col}(A)$

If we are interested in row vectors of A that form a basis for $\text{row}(A)$:

$$A \rightarrow A^T \xrightarrow[\text{ops}]{\text{(elem.) row}} R \rightarrow \text{Columns of } R \text{ that form a basis correspond to columns of } A^T, \text{ which are rows of } A.$$

#15 Find a basis for the subspace of R^4 that is spanned by the given vectors.

$(1, 1, 0, 0), (0, 0, 1, 1), (-2, 0, 2, 2), (0, -3, 0, 3)$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The given set is the desired basis.

$$2\vec{w}_1 - \vec{w}_2 : 2\vec{v}_1 - \vec{v}_2 = (2, -2, 10, 4) + (2, -3, -1, 0) \\ = (4, -5, 9, 4)$$

#17 Find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors.

$\vec{v}_1 = (1, -1, 5, 2), \vec{v}_2 = (-2, 3, 1, 0), \vec{v}_3 = (4, -5, 9, 4), \vec{v}_4 = (0, 4, 2, -3),$
 $\vec{v}_5 = (-7, 18, 2, -8)$

$$\begin{array}{cccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \\ \begin{bmatrix} 1 & -2 & 4 & 0 & -7 \\ -1 & 3 & -5 & 4 & 18 \\ 5 & 1 & 9 & 2 & 2 \\ 2 & 0 & 4 & -3 & -8 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 0 & 2 & 8 & 15 \\ 0 & 1 & -1 & 4 & 11 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & & \end{array}$$

Row echelon form gives us leading 1s

Basis is $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$.

Continuing the row reduction:

$$\begin{array}{ccccc} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 & \vec{w}_4 & \vec{w}_5 \\ \left[\begin{array}{ccccc} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

reduced row echelon form — gives dependency equations

We can see that $\vec{w}_3 = 2\vec{w}_1 - \vec{w}_2$ and $\vec{w}_5 = -\vec{w}_1 + 3\vec{w}_2 + 2\vec{w}_4$

Row operations preserve dependency equations.

Thus, $\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$ and $\vec{v}_5 = -\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_4$

Definition: An equation that expresses a column vector of a matrix A that does not contain a pivot as a linear combination of column vectors that contain pivots is a **dependency equation**.

#18 Find a basis for the row space of A that consists entirely of row vectors of A .

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Corresponding rows of A are

$$\{[1 \ 4 \ 5 \ 2], [2 \ 1 \ 3 \ 0]\}.$$

$A \longrightarrow R$

Basis for
col(A)

Basis for null(A)

Basis for row(A)